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NOTATION

$C(s)$	complex amplification
$G_o(s)$	transfer function of the open system
$G(s)$	transfer function of the control system
K	value of the amplification
M	degree of the function G_o
m	degree of the polynomial in the numerator
n	degree of the polynomial in the denominator
$N_o(s)$	denominator of the transfer function of the open system
n_i	the ith zero of G_o
p_i	the ith pole of G_o
R_o	the residue of G at the point p_o
r_i	value of $s - s_i$
s	complex variable
$s(t)$	parameter representation of a plane curve
V	real amplification
$x(t)$	parametric representation of a space curve
$Z_o(s)$	numerator of the transfer function
z_i	the ith branch point
α_i	multiplicity of the root s_i
K	curvature of the root-loci of constant phase
$\mu_i - 1$	multiplicity of the branch point z_i
$\varphi_i \varphi_o$	phase of the amplification
Ψ, Ψ_E	direction angle of the root-loci of constant phase
Ψ_U	direction angle of the root-loci of constant value

PROPERTIES OF GENERALIZED ROOT-LOCI

Grübel and L. Schmieder

ABSTRACT: In this paper generalized root-loci are introduced, which are generated by mapping the polar coordinate lines of the C-plane into the s-plane. C(s) is the system gain which is in its general form complex. By means of the theory of complex functions it is possible to easily derive the essential properties of the root-loci. Furthermore Schwarz's reflection principle can be applied to show that straight lines and circles can be root-loci if and only if the poles as well as the zeros lie symmetrically to these straight lines and circles. From this the symmetrical properties of the root-loci are derived. In addition to the known methods for constructing the root-loci graphically, a method is derived for the stepwise computation of root-loci. The step size is estimated by means of the curvature. The formula of Ur, giving the curvature in regular and simple breakaway points, is generalized. An illustrating example is given.

1. Introduction

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The root-loci introduced by W. R. Evans [1] have acquired great importance for the analysis and synthesis of control loops. As an example, let us look at the transfer function G(s) of the system represented in Fig. 1.

$$G(s) = \frac{G_0(s)}{1/V + G_0(s)} \cdot \qquad (1.1)$$

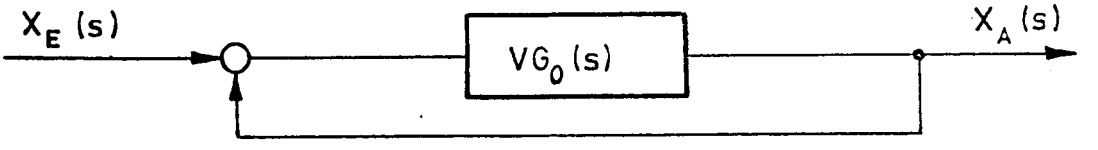

Figure 1. Block Diagram of a Control Loop With Unit Feedback.

Here, $VG_0(s)$ is the transfer function of the open system and V is the (real) amplification. If we change the amplification, we also change the poles of the transfer function G(s), but

*Numbers in the margin indicate pagination in the foreign text.

the zeros remain fixed. If we trace the path of the poles in the complex plane, we get the root loci introduced by Evans. Some authors admit also a complex amplification (for example, see [2]). In such a case, we need to replace the real parameter V with the complex parameter C. If we change only the magnitude of C but hold the phase constant, we get the so-called phase loci [2]. These play a role in the investigation of nonlinear systems and systems with dead time or with partitioned parameters. In the case of Evans' root-loci, the phase is equal to π.

From a function-theoretic standpoint, the situation is described as follows: A single-valued analytic function $G_0(s)$ is defined in a domain $\mathfrak{D}$ of the s-plane. With the aid of it, we then define a function

$$G(s,C) = \frac{G_0(s)}{G_0(s) - 1/C} \tag{1.2}$$

For a fixed value C_0 such that $1/C_0$ belongs to the range $\mathfrak{L}^*$ of the function $G_0(s)$, every value s_0 such that $G_0(s_0)$ is a pole of the function $G(s, C_0)$. If, in addition, the first derivative $G_0'(s_0) \neq 0$ for a finite value of $G_0(s_0)$, we have a simple pole of $G(s, C_0)$, and the residue of the function $F(s, C_0)$ at this point is /8

$$R_0 = [\text{Res } G(s,C_0)]_{s=s_0} = \frac{G_0(s_0)}{G_0'(s_0)}, \tag{1.3}$$

because it follows from the Taylor expansions of the numerator and denominator of the function $G(s, C_0)$ at the point s_0 that

$$G(s,C_0) = \frac{G_0(s_0) + G_0'(s_0)(s-s_0) + \ldots}{G_0(s_0) - 1/C_0 + G_0'(s_0)(s-s_0) + \ldots} =$$

$$= \frac{G_0(s_0)}{G_0'(s_0)} \frac{1}{s-s_0} + \text{Power series in } s-s_0. \tag{1.4}$$

Now, by setting s equal to the pole of the function G(s, C), we assign to each $s \in \mathfrak{D}$ the parameter value $C = 1/G_0(s)$. This defines a single-valued mapping of the region $\mathfrak{D}$ of the s-plane onto a region $\mathfrak{L}^*$ of the C-plane. However, this mapping is not in general one-to-one. On the other hand, we can partition the region $\mathfrak{D}$ into fundamental regions in which the mapping is one-to-one. Then, a curve Γ^* in the region $\mathfrak{L}^*$ of the C-plane has an image Γ in each fundamental region of the s-plane. We call the totality of these image curves a general root-locus.[1]

[1] The possibility of such a definition was indicated already in [3].

Now, let s(t) denote the parametric representation of a curve Γ (with real parameter t) in a fundamental region $\mathfrak{G}$ of the s-plane. Then, $1/G_o(s(t))$ is the parametric representation of the corresponding image curve Γ^* in the C-plane. If s_o is a simple pole of the function $G(s, C_o)$, then, for the tangent vectors, we have the relation

$$\left(\frac{dC}{dt}\right)_o^{\Gamma^*} = - \frac{G'_o(s_o)}{G_o^2(s_o)} \left(\frac{ds}{dt}\right)_o^{\Gamma}, \tag{1.5}$$

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where $\left(\frac{ds}{dt}\right)_o^{\Gamma}$ represents the tangent vector of the curve Γ at the point s_o and $\left(\frac{dC}{dt}\right)_o^{\Gamma^*}$ represents the tangent vector of the curve Γ^* at the point C_o. When we substitute equation (1.3), we get

$$\left(\frac{ds}{dt}\right)_o^{\Gamma} = - R_o \left(\frac{d \log C}{dt}\right)_o^{\Gamma^*} . \tag{1.6}$$

We now choose the polar coordinate lines as curves Γ^* in the C-plane by setting

$$C = Ke^{j\varphi} \Rightarrow \log C = \ln K + j\varphi \tag{1.7}$$

For the lines φ = const (root-loci of constant phase), we can make the substitution t = = ln K in Eq. (1.6). We get

$$\frac{ds}{d \ln K} = - R_o . \tag{1.8}$$

Accordingly, we get the following relation for the root-locus curves of constant magnitude (K = const) with $t = \varphi$:

$$\frac{ds}{d\varphi} = - jR_o . \tag{1.9}$$

Equation (1.9) says that the value of the residue R_o of the function G(s, C) at a regular point s_o is proportional to the distance between the root-loci of constant phase, while we can read the argument of the residue from Eq. (1.8) (see [4]).

2. The Most Important Properties of Root-Loci

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2.1. Root-Loci and Impulse Response

In what follows, we shall confine ourselves to root loci of constant phase and constant value of C. Furthermore, the function $G_o(s)$ is assumed to be a rational function.

Suppose that its product representation is [2]

$$G_o(s) = \frac{Z_o(s)}{N_o(s)} = \frac{(s-n_1)\ldots(s-n_n)}{(s-p_1)\ldots(s-p_m)} = \prod_i (s-s_i)^{\alpha_i} \quad . \tag{2.1}$$

The quantity $|\alpha_i|$ yields the multiplicity of the zero $s_i = n_i$ (for $\alpha_i > 0$) or of the pole $s_i = p_i$ (for $\alpha_i < 0$). The degree of the numerator (resp. denominator) polynomial is n(resp. m). We call

$$M = \mathrm{Max}(n, m) \tag{2.2}$$

the degree of the function $G_o(s)$. The assignment "let s denote a pole of the function G(s, C)" then becomes (see Eq. (1.2))

$$C(s) = K(s)\, e^{i\varphi_o(s)} = 1/G_o(s) = N_o(s)/Z_o(s) \qquad \text{or}$$

$$\log C(s) = \ln K(s) + i\varphi_o(s) = -\sum_i \alpha_i \ln r_i - i \sum_i \alpha_i \varphi_i \tag{2.3}$$

where[3]

$$r_i = |s-s_i| \,, \qquad \varphi_i = \arg(s-s_i) \quad .$$

For a fixed C_o, the equation (see first of Eqs. (2.3))

$$C_o Z_o(s) - N_o(s) = 0 \tag{2.4}$$

has in general M distinct solutions. Therefore, there are exactly M fundamental regions /11 and M distinct branches of the root-locus. The tangent vectors to the root-loci are given by Eqs. (1.8) and (1.9). The residue Eq. (1.3) is referred to in [4] as the pole sensitivy. Since, in accordance with the remarks at the end of the preceding section, this can be read from the root-loci, it is also possible in this way to derive the impulse transfer function of the closed control loop (see Fig. 1). It is obtained from $G(s, C_o)$ by means of the inverse Laplace transformation:

$$g(t) = \sum_1^M R_i e^{S_i t} \tag{2.5}$$

where S_i is a simple pole of the function $G(s, C_o)$ and R_i is the residue of $G(s, C_o)$ at the point S_i.

---

[2] A constant factor by which $G_o(s)$ may be multiplied is included in the amplification C.

[3] In the present article, the roots of the function $G_o(s)$ are denoted by s_j. They are roots of the equation $Z_o(s) = 0$ or $N_o(s) = 0$.

2.2 The Analogy With the Electrostatic Field

It follows from the real part of the second of Eqs. (2.3) that In K can be interpreted as the electrostatic potential of infinitely long rods with linear charge density $2\pi\varepsilon_o\alpha_i$ [coulomb/m] that are perpendicular to the s-plane and penetrate this plane at the zeros of the function $G_o(s)$. The lines of force of the electric field correspond to the root-loci of constant phase; the equipotential curves correspond to root-loci of constant magnitude. The tangent angles to these curves are, in accordance with Eqs. (1.8) and (1.9), equal to

$$\psi_E = -\pi + \arg G_o - \arg G'_o , \qquad \psi_U = \psi_E + \pi/2 . \tag{2.6}$$

The analogy with the electrostatic field is used in the calculation of the root-loci, which rests on the principle of the electrolytic tank.

2.3. The Behavior of Root-Loci at Singular Points. Asymptotes

The value of C(s) becomes infinite as we approach a zero of order α_i of the function $G_o(s)$. The behavior of the root loci of constant phase is determined by the term of the Laurent expansion

$$C(s) = A_{-\alpha_i}/(s-n_i)^{\alpha_i} + \ldots \tag{2.7}$$ /12

with the highest negative exponent. With regard to Eq. (2.1) and the first of Eqs. (2.3), we get the coefficient

$$A_{-\alpha_i} = (s-n_i)^{\alpha_i}/G_o(s) = \prod_{k\neq i}{}' (s-s_k)^{-\alpha_k} . \tag{2.8}$$

$\lim (-s + n_i)$ is the tangent vector to a root locus that runs into the zero $s = n_i$. With $\arg G_o = -\varphi_o \bmod 2\pi = -\arg C$ (see Eqs. (2.3)) we get the direction angles of the root loci that run starlike into the zero from Eq. (2.8)

$$\psi(n_i) = -[\varphi_o + \sum_{k\neq i} \varphi_k\alpha_k]/\alpha_i + \pi + 2\pi\nu/\alpha_i \tag{2.9}$$

with $\nu = 0, 1, \ldots, \alpha_i - 1$. By means of analogous reasoning, we derive from the Taylor expansion about a pole p_i of order α_i the directions of the root-loci issuing from it:

$$\psi(p_i) = [\varphi_o + \sum_{k \neq i} \varphi_k \alpha_k] / |\alpha_i| + 2\pi\nu/|\alpha_i| \tag{2.10}$$

where $\nu = 0, 1, \ldots, \alpha_i - 1$. At a branch point z_i of order $\mu_i - 1$, we must distinguish between root-loci entering the point and issuing from it. We get the direction angles

$$\begin{aligned} \psi(z_i)_{\text{issuing}} &= -[\pi + \arg(\log G_o^{(\mu_i)}) + 2\pi\nu]/\mu_i \\ \psi(z_i)_{\text{entering}} &= \psi(z_i)_{\text{issuing}} + \pi/\mu_i + \pi \end{aligned} \tag{2.11}$$

where $\nu = 0, 1, \ldots, \mu_i - 1$. The μ_ith derivative is especially easy to calculate at the branch point because $G_o' = G_o'' = \ldots G_o^{(\mu_i - 1)} = 0$:

$$(\log G_o(s))^{(\mu_i)}_{s=z_i} = G_o^{(\mu_i)}(z_i)/G_o(z_i) = (-1)^{\mu_i - 1}(\mu_i - 1)! \sum_{k=1}^{M} \frac{\alpha_k}{(z_i - s_k)^{\mu_i}} . \tag{2.12}$$

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At a branch point of order $\mu_i - 1$, there are as many entering and issuing root-loci of constant magnitude as there are of constant phase whose directions differ from each other by an angle of $\pi/2\mu_i$ since the right angle at which these curves intersect at a regular point (see Eqs. (1.8) and (1.9)) is reduced at the branch point z_i to $1/\mu_i$ its former value.

In the analogous electrostatic field, all charged rods work together at a great distance from the zeros like a single rod with charge density $(2\pi\epsilon_o)k$ located at the common center of gravity of all roots. Therefore, the lines of force at a great distance from the center of gravity approach the rays which issue starlike from the center of gravity and which have directions given by

$$\psi_{\text{Asymptote}} = [\varphi_o + 2\pi\nu]/k \tag{2.13}$$

where $k = m - n$ and $\nu = 0, 1, \ldots, |k| - 1$. If $k = 0$, there are no asymptotes since we then have a dipole or a multipole field. For literature on this section, see [5].

3. The Symmetry Properties of Root-Loci

The function-theoretic discussion made in section 1 enables us to apply Schwarz' reflection principle and with it to derive a number of new properties of root-loci. We begin by proving the theorem:

A straight line or a circle is a root-locus of constant phase if and only if both the poles and the zeros of $G_o(s)$ come in pairs that are symmetric about that line or circle.

One can see immediately that this condition is necessary because the poles and the zeros in the C-plane lie on the rays issuing from the origin; consequently, they must lie symmetric in the s-plane to the root-loci of constant phase. That this symmetry condition is also sufficient, we shall prove first for a straight line by calculating the phase of the root-locus lying on it. To this end, we equip the line being investigated with a direction arrow (See Fig. 2) and examine two simple roots P_1 and $\overline{P}_1$ that are symmetric about it. For these roots we have $\varphi_1 + \overline{\varphi}_1 = 2\gamma + \varepsilon_1 + \overline{\varepsilon}_1 = 2\gamma$, where γ is the direction angle of the line. If a simple root lies on the line before the test point, then $\varphi = \gamma + \pi$. In accordance with Eq. (2.3), the phase condition

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$$\varphi_o = k\gamma + \pi \sum_i \alpha_i \qquad \text{with } k = m-n \tag{3.1}$$

must be satisfied, where the summation is over all roots on the straight line before the test point. Since the symmetry condition is satisfied in general and a fortiori for the entire line, this entire line is a root locus. Now a circle can always be mapped by means of a bilinear transformation into a circle and reflection symmetry is conserved under this transformation. Thus, the symmetry condition is always sufficient for circles and the above theorem is proven.

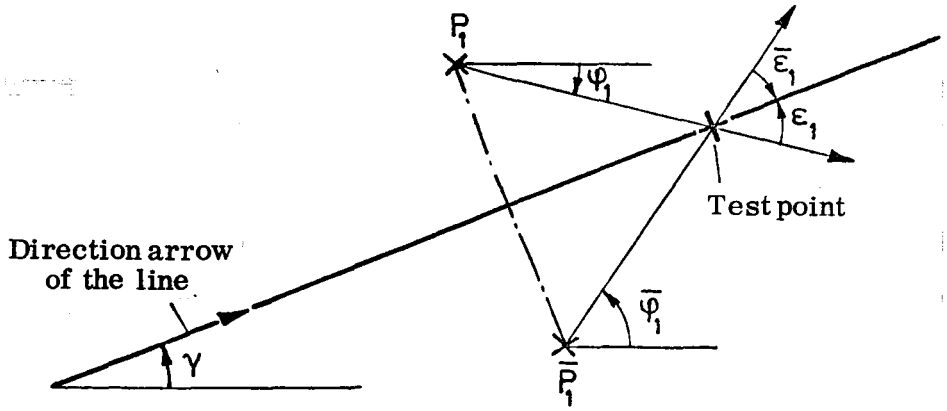

Figure 2. Drawing Illustrating the Proof of the Sufficiency of the Symmetry Condition

From the reflection principle and Eq. (3.1), we get several other theorems, which, like the preceding, also hold for both lines and circles. /15

1. Only two distinct root-loci that differ by phase π and can meet only at poles or zeros of odd order can lie on a single straight line. Passing through a zero of order α cause a phase jump of $\alpha\pi$.

2. Different branches of the same root locus can meet on a straight line only at points that are branch points or roots of even order. (In the latter case, the

phase jumps by a multiple of 2π.) A root-locus can leave a straight line only in branch points or in multiple roots.

3. The real axis is always a root locus of phase 0 or π. Those parts of the real axis on whose right side lie an odd number of roots have phase π.

4. All root loci of phase $\varphi \pm \Delta\varphi$ are symmetric to a straight root locus with phase φ. In particular, all branches of the root-locus φ are symmetric about this line. All branches of the root-locus with phase 0 and φ are symmetric about the real axis.

From the symmetry condition we get certain rules for the existence of straight lines or circles as root-loci. Except for the real axis, a straight line can be a root-locus only when the centers of gravity of the poles and zeros coincide. The calculation of the center of gravity from the coefficients in the polynomial is done with the aid of Viets' root theorem:

$$s^n + a_1 s^{n-1} + \ldots a_n = s^n - s^{n-1} \sum_1^n s_i + \ldots + (-1)^n s_1 s_2 \ldots s_n . \qquad (3.2)$$

The coefficient a_1 divided by n yields the center of gravity.

For the appearance of circles as root loci, we have more stringent restrictions. /16
We distinguish the cases $k \neq 0$ and $k = 0$, where $k = m - n$:

1. $k \neq 0$.
   Then, there is a root of multiplicity k at ∞. Therefore, in view of the necessary reflection symmetry, the center of the circle must always coincide with a root of multiplicity k. Therefore, there can only be a circle whose center lies on the real axis.

2. $K = 0$.
   (a) We have only one zero and one pole. Then the set of root loci is the pencil of curves through these two points. The centers of the circles are the points of intersection of the middle perpendicular with those lines that pass through the roots of multiplicity α at an angle $\pi - (\varphi_o + 2\pi\nu)/\alpha$.

   (b) In the other cases, there can again be only one circle as root locus which must be symmetric with respect to the real axis.

Proof: Pairs of points that are mirror images of each other about two circles must lie on the line connecting the centers of the circles. One can now show easily that there exist only a single pair of points that consist either of two poles or of two zeros. But to satisfy the condition $k = 0$, there must be at least two pairs of roots of opposite multiplicity.

The phase of the root-loci that lie on circles symmetric about the real axis can be only 0 or π because, on the one hand, such a circle contains two points of the real axis and, on the other, at most two distinct phases can exist which must differ by π.

4. Example of a Root-Locus Field

The transfer function

$$G_o(s) = \frac{s^2 + 6s + 9}{s^4 + 4s^3 + 5s^2 - 2s - 8} \tag{4.1}$$

has four simple poles and a single double zero: /17

$$p_1 = 1; \quad p_2 = -2; \quad p_{3,4} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}; \quad n_{1,2} = -3 \quad .$$

For $k = 2$, there are two zeros at ∞. Therefore, every root locus of constant phase has two asymptotes with direction angles

$$\psi_{Asympt} = \frac{\varphi_o}{2} + \nu\pi \quad \text{with } \nu = 0, 1 \quad .$$

The double zero is the center of a circle that is a root locus. Its radius is 2 (see Fig. 3). The intersections of this circle with the real axis are branch points whose data are shown in Table 1. $\mu - 1$ is the multiplicity of the branch point and φ_o is the phase of the corresponding root-locus.

TABLE 1. Data on the Branch Points /18

	s	μ	φ_o	$\log G_o$	$(\log G_o)'$	$(\ )''$	$(\ )'''$	$(\ )^{IV}$	$(\ )^{V}$
z_1	-1	4	π	$i\pi$	0	0	0	6	-30
z_2	-5	2	0	$-\log 63$	0				

From the formulas of section 2.3, we get the direction angles at the singular points as shown in Table 2.

The reader is now referred to the representation in Fig. 3. Note the symmetry of the locus field with respect to the circle with center $n_{1,2}$.

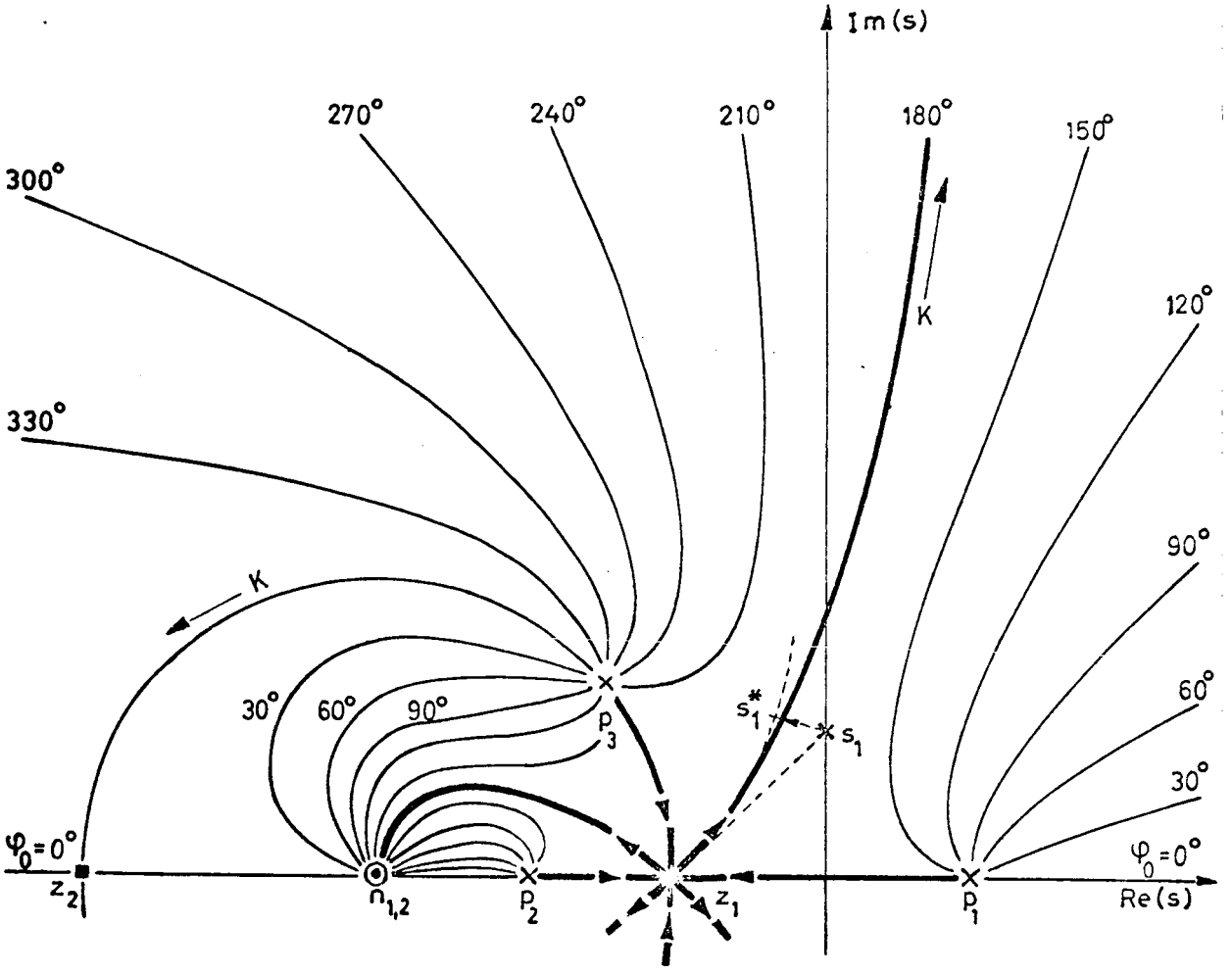

Figure 3. Root Locus Field for the Transfer Function $G_o(s) = \dfrac{s^2 + 6s + 9}{s^4 + 4s^3 + 5s^2 - 2s - 8}$

TABLE 2. Direction Angles at the Singular Points

i =	1	2	3	4
$\psi(p_i)$	φ_o	$\varphi_o + \pi$	$\varphi_o + 48,5°$	$\varphi_o - 48,5°$
$\psi(n_i)$	$\varphi_o/2 + (n+1)\pi$ with $n=0, 1$			
$\psi(z_i)_{ent}$	$n\pi/2$	$\pi/2+n\pi$		
$\psi(z_i)_{iss}$	$\pi/4+n\pi/2$	$n\pi$		
	with $n=0,\ldots3$	with $n=0, 1$		

5. The Curvature of Root-Loci

Let x(t) denote the position vector of a real space curve and let $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second derivatives of this position vector with respect to t. Then, we have

$$\varkappa \underline{b} = \frac{\dot{\underline{x}} \times \ddot{\underline{x}}}{|\dot{\underline{x}}|^3} \tag{5.1}$$

where $\varkappa$ is the principal curvature of the curve and b is the binormal vector of the curve. In the case of plane curves, b is perpendicular to the plane in which the curve lies. In this case, x(t) and $\dot{x}(t)$ can be expressed by a complex variable s(t) and its derivative $\dot{s}(t)$. Then, the cross product $\dot{x} \times \ddot{x}$ becomes Im $(\dot{s}\,\ddot{s})$. Equation (5.1) then becomes /19

$$\varkappa = \mathrm{Im}\,(\ddot{s}/(\dot{s}|\dot{s}|)) \quad . \tag{5.2}$$

We now expand an analytic function F(s) in the neighborhood of a branch point z of order $\mu - 1$ in a power series about $\Delta s = s - z$:

$$F(s) - F(z) = \Delta F(s) = \frac{F^{(\mu)}(z)}{\mu!} \Delta s^{\mu} + \frac{F^{(\mu+1)}(z)}{(\mu+1)!} \Delta s^{\mu+1} + \ldots \tag{5.3}$$

The derivatives of lower orders are all equal to zero. For the multiple-valued inverse function s(F), we have the series expansion

$$\Delta s = \dot{s} \Delta F^{1/\mu} + \frac{\ddot{s}}{2} \Delta F^{2/\mu} + \ldots \tag{5.4}$$

Here, $\dot{s}$ denotes the derivative with respect to $\Delta F^{1/\mu}$. If we substitute this expression for Δs into Eq. (5.3), we get

$$\Delta F(s) = \frac{F^{(\mu)}(z)}{\mu!} \left[\dot{s}^{\mu} \Delta F + \mu \dot{s}^{\mu-1} \frac{\ddot{s}}{2} (\Delta F)^{\frac{\mu+1}{\mu}} + \ldots \right] +$$

$$+ \frac{F^{(\mu+1)}(z)}{(\mu+1)!} \left[\dot{s}^{\mu+1} (\Delta F)^{\frac{\mu+1}{\mu}} + \ldots \right] + \quad .$$

Let us transpose $\Delta F(s)$ to the right side and equate coefficients of equal powers of ΔF. This yields

$$\dot{s}^{\mu} = \frac{\mu!}{F^{(\mu)}} \; : \qquad \ddot{s} = - \frac{2}{\mu(\mu+1)} \frac{F^{(\mu+1)}}{F^{(\mu)}} \dot{s}^2 \quad . \tag{5.5}$$

Let us now consider whether these expressions may be substituted into Eq. (5.2) (in that equation, ṡ denoted the drivative with respect to a real parameter). For example, if we hold the imaginary part fixed in F = x + jy, the expression ds/dF denotes the derivative with respect to a real parameter x. With complex numbers, however, the value of the derivative is independent of the direction with respect to which the differentiation is carried out. Therefore, the coefficients of the power series (5.4) is independent of our choice of ΔF. For this reason, the expressions (5.5) can be substituted into the curvature formula (5.2), where we take $F(s) = \log C = -\log G_o$: /20

$$\varkappa = - \operatorname{Im} \frac{2}{\mu(\mu+1)} \frac{\log G_o^{(\mu+1)}}{\log G_o^{(\mu)}} e^{j\psi} . \tag{5.6}$$

ψ is the argument of ṡ, which needs to be calculated in accordance with Eq. (2.11). Only in ψ is the special choice of the curve s(F) or the parameter t changed. For the special cases $\mu = 1$ and 2, the curvature formula has already been given by Ur [4].

Only the branches of a single root-locus with phase φ_z pass through the branch point. The smaller the phase difference $\varphi - \varphi_z$, the closer the root-loci of neighboring phase approach them. We obtain quantitative results with the aid of the Taylor series

$$G_o(s) = G_o(z) + \Delta s \frac{G_o^{(\mu)}(z)}{\mu!} + \ldots \tag{5.7}$$

and the derivatives

$$G'_o(s) = \frac{(\Delta s)^{\mu-1}}{(\mu-1)!} G_o^{(\mu)}(z) + \ldots$$

$$G''_o(s) = \frac{(\Delta s)^{\mu-2}}{(\mu-2)!} G_o^{(\mu)}(z) + \ldots \tag{5.8}$$

The expression

$$\left[\frac{(\log G_o)''}{(\log G_o)'}\right](s) = \left[G''_o/G'_o - G'_o/G_o\right](s) =$$

$$= \frac{\mu-1}{\Delta s} - \frac{(\Delta s)^{\mu-1}}{(\mu-1)!} \frac{G_o^{(\mu)}(z)}{G_o(z) + (\Delta s)^{\mu} G_o^{(\mu)}(z)/\mu!} , \tag{5.9}$$

multiplied by $e^{j\psi}$ gives the curvature of the root locus in the neighborhood of the branch point. We let this range over a small circle $\Delta s = re^{j\vartheta}$. The phases of the root-loci that are cut in this way are only slightly different from $\varphi_z = -\arg G_o(z)$ since the second term in Eq. (5.7) is small as compared with the first. From Eq. (5.8) we obtain by use of Eq. (2.6) the direction angle of the root locus of constant phase: /21

$$\psi = -\pi - \varphi_z - \psi_z - (\mu-1)\vartheta \tag{5.10}$$

where $\psi_z = \arg G_o^{(\mu)}(z)$. The directions of the root-loci accordingly range $\mu - 1$ times over a complete circle when one encircles the branch point once.

We obtain the curvatures of the root-loci that range over the neighborhood of a branch point by substituting Eqs. (5.10) and (5.9) into the curvature formulas (5.6), where we need to consider only the first term on the right side of Eq. (5.9), namely,

$$\frac{\mu-1}{\Delta s} = \frac{\mu-1}{r} e^{-j\vartheta} ,$$

We obtain

$$\varkappa \approx \frac{\mu-1}{r} \operatorname{Im}\, e^{j(-\varphi_z-\psi_z-\mu\vartheta)} . \tag{5.11}$$

Close to zero, one encirclement of the branch point increases the curvature 2μ times. Each time it assumes the maximum $(\mu - 1)/r$.

6. A Procedure for Numerical Determination of the Curve Path

It is assumed that the poles, zeros, and branch points are known. Then, the regular points of the root loci can be obtained by the following procedure: Beginning with a known point s_o, we calculate the next point by means of

$$s = s_o + e^{j\psi} |\Delta s| \left(1 + \frac{j\varkappa |\Delta s|}{2}\right) , \tag{6.1}$$

where the step is determined by giving a fixed value

$$\tau_o = \frac{|\varkappa \Delta s|}{2} < 1 \tag{6.2}$$

It is necessary to give an upper bound for the step, for example, the distance to the next root or to the next branch point. The improvement of the new point s_1 must, on the basis of the Taylor expansion, satisfy the equation /22

$$\log G_o(s_1 + \Delta s) = \ln |G_o| + j \arg G_o + \Delta s \frac{G'_o(s_1)}{G_o(s_1)} =$$

$$= -\ln K_{soll} - j\varphi_{o\,soll} \tag{6.3}$$

In the improvement process, either the magnitude or the phase of C is held fast. In the first case (when one moves along a root locus of constant magnitude), we need to set $\Delta s = |\Delta s| e^{j\psi_U}$ into Eq. (6.3) (the ψ_U is in accordance with Eq. (2.6)). In this case, only the imaginary part of this equation is of significance:

$$j \arg G_o + \left| \Delta s \frac{G'_o}{G_o}(s_1) \right| e^{-j\pi} = -j\varphi_{o\,soll}$$

or

$$|\Delta s| = \left| \frac{G_o}{G'_o}(s_1) \right| (\arg G_o + \varphi_{o\,soll}) \quad .$$

The sought change in s, which is necessary to get to the correct phase φ_o is therefore

$$\Delta s_{\varphi_o} = (\varphi_{o\,soll} + \arg G_o) \left| \frac{G_o}{G'_o} \right| e^{j\psi_U} \quad . \tag{6.4}$$

In an analogous manner, we find by proceeding along a root-locus of constant phase

$$\Delta s_K = (\ln K_{soll} + \ln |G_o|) \left| \frac{G_o}{G'_o} \right| e^{j\psi_E} \quad . \tag{6.5}$$

As an example, let us calculate, for the branch point (-1) in the root locus field shown in Fig. 3, a neighboring point to a root locus $\varphi_o = \pi$. The curvatures at the branch point are (see Table 1).

$$\varkappa_E = 1/2, \sqrt{2}/4 \, .$$

As we proceed along the tangent with step $\tau_o = 1/4$, we reach the point s_1 on the imaginary axis (see Fig. 3) with arg $G_o(s_1) = -167°$: $\psi_U = 168.7°$ and with improvement $\Delta s_{\varphi_o} = 0.334e^{j\psi} U$. The phase error at the improved point $s_1 = -0.328 + 1.066j$ is then 2.1°. We attain the same accuracy by using the quadratic formula (6.1) in the first step. /23

The calculations of this section could be obtained much more quickly since there exists an algol program for digital calculating machines that operates in accordance with the procedure described here [6].

CONCLUSIONS

In the present work, the properties of root-loci are investigated by using the tools of function theory. Another possibility is the examination of root-loci as algebraic curves. This method could be advantage especially for systems of low order, which are of particular interest as basic configurations. As the order **increases, the setting up and solving of the algebraic equations becomes** rapidly more complicated. Up to now, only root-loci of phase 0 or π have been treated in mathematical literature with the aid of these methods. (See the work of Föllinger [7], in which this method was presented for the first time.)

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